Accurate Shooting from Fast Moving Platforms

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Abstract

I present the derivation and analysis of equations for the determination of precise reverse lead and angular offset for effective single shot accuracy when shooting from a fast moving platform (aircraft, helicopter, ground vehicle, boat, etc.) at a stationary target. I include a highly effective method for applying the theoretical result for real-world training and application, with verification of effectiveness and an error analysis.

Introduction

When the need arises, shooters must be able to provide accurate fire from a fast moving platform. In this regard, "accurate" is a relative term, as common moving platforms inherently vibrate, may be affected by wind gusts or seas, and can be maneuvered by pilots or drivers without warning. In spite of all of this, a well-trained operator can achieve excellent single shot close-in results through the use of the ballistics considerations presented in this paper. For purposes of example, we assume the moving platform is a helicopter moving in a straight line past a stationary target at speeds in excess of 50 knots. The actual platform could be quite different, of course, but the results given in this paper apply over a wide range of platforms, speeds and choices of weapons. The shooter must employ a "reverse lead" or a "lag". That is, the shooter must aim a specified distance (offset from the target) in the direction opposite to the direction of platform travel. The mathematical construct for the computation of target reverse lead that follows is easier to understand with reference to Figure 1, below.

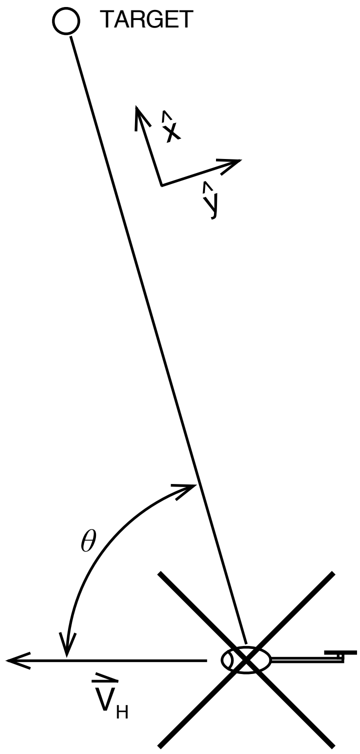

Figure 1: Plan View of Helicopter Pass and Shot

There is an underlying assumption inherent to this analysis: The moving platform (a helicopter in this case) is traversing a straight-line path with respect to the stationary target and its speed is close to constant. We omit the proof that the result is still correct for a circular orbit, though the "virtual target" revolves with the platform.

θ is the angle between the line defining the direction of vehicle travel and the intended target. When θ is 90°, the range from the moving platform to the target is at a minimum, and we call that range X_0. In the present derivation, we shall limit X_0 to 300 yards or less. It is certainly possible to extend beyond this limit, but accurate solution quickly requires numerical integration techniques rather than the algebraic solution presented here.

All units below are in pounds, feet and seconds and various combinations thereof.

The coordinate system is "Shooter Centered". In this system,

$\hat{x}$	unit vector from the muzzle toward the target
$\hat{y}$	unit vector normal to $\hat{x}$ to the shooter's right side

The origin is at the center of the muzzle. Note that, in this system, the x-axis always points from the muzzle toward the target. We designate X as the (time and vehicle position dependent) instantaneous range from the muzzle to the target.

The helicopter's ground speed vector is $\vec{V}_H$ with magnitude V_H and the angle of the shot direction away from the helicopter's ground-track direction θ. This gives us

$$\vec{V}_H = V_H \cos\theta\, \hat{x} - V_H \sin\theta\, \hat{y} \tag{1}$$

The muzzle velocity is $\vec{V}_M = V_M\, \hat{x}$ where V_M is the muzzle speed. The initial vector ground velocity of the bullet, $\vec{V}_B(0)$, is the vector sum of the bullet muzzle velocity and the helicopter velocity.

$$\vec{V}_B(0) = \vec{V}_M + \vec{V}_H = (V_M + V_H \cos\theta)\, \hat{x} + V_H \sin\theta\, \hat{y} \tag{2}$$

$V_B(0)$ is the magnitude of the bullet velocity relative to the ground.

Because we are limiting the shooting range to within 300 yards, we need only be concerned with the muzzle velocity and the ballistic coefficient appropriate to the muzzle velocity. Because the range is limited to around 300 yards, we can estimate the final velocity of the bullet at the target using a single ballistic coefficient (at or near the muzzle velocity) and a single value of the appropriate Siacci "G" function (also evaluated at the muzzle velocity). If we call the standard, published ballistic coefficient BC_0, we can compute the initial drag coefficient ([McCoy99], p. 90):

$$\hat{C}_d = \frac{\rho \pi G(V_B(0)/c_s)}{1152 BC_0} \tag{3}$$

where $G(V_B(0)/c_s)$ is the Siacci drag function evaluated at the bullet muzzle velocity expressed as a Mach number (that is, divided by the speed of sound, c_s). Using $c_s = 1120$ feet per second (at 59 degrees Fahrenheit in dry air at sea level) is good enough for our purposes. V_M can be safely substituted for $V_B(0)$ for the current purpose without sacrificing accuracy in this equation. ρ is the density of the air. Again the sea level, dry, 59 degree value, 0.07647 pounds per cubic foot suffices. If the shooter is at a very different environment (say, high in mountainous territory) then this value should be modified as necessary.

Now, the range to the target when we are shooting at an angle different than $\theta = 90°$ (right angles to the moving platform direction of motion) is

$$X = \frac{X_0}{\sin\theta} \tag{4}$$

Next, we estimate the final bullet velocity vector at the range of the target (i.e., at impact). This is given by a vector version of ([McCoy99], p. 92):

$$\begin{aligned} \vec{V}_B(X) &= \vec{V}_B(0)\, e^{-\hat{C}_d X} \\ &= (V_M + V_H \cos\theta)\, e^{-\hat{C}_d \frac{X_0}{\sin\theta}}\, \hat{x} + V_H \sin\theta\, e^{-\hat{C}_d \frac{X_0}{\sin\theta}}\, \hat{y} \end{aligned} \tag{5}$$

This equation for the final velocity allows fairly accurate estimation of the time of flight(also from [McCoy99], p.92):

$$t = \frac{X\left(\frac{V_B(0)}{V_B(X)} - 1\right)}{V_B(X)\ln\left(\frac{V_B(0)}{V_B(X)}\right)} \tag{6}$$

$V_B(X)$ is the magnitude of $\vec{V}_B(X)$ from equation 5.

Note that the $\hat{x}$ component of $\vec{V}_B$ in equation 5 is much greater than its $\hat{y}$ component. In equation 6 above, we ignore the smaller $\hat{y}$ component of $\vec{V}_B$ and then simplify algebraically. The result is that the time of flight is accurately estimated by

$$t \approx \frac{e^{\hat{C}_d X} - 1}{V_B(X)\hat{C}_d} \tag{7}$$

or

$$t \approx \frac{e^{\hat{C}_d \frac{X_0}{\sin\theta}} - 1}{V_M\, e^{-\hat{C}_d \frac{X_0}{\sin\theta}}\hat{C}_d} \tag{8}$$

This close approximation is sufficient for our purposes.

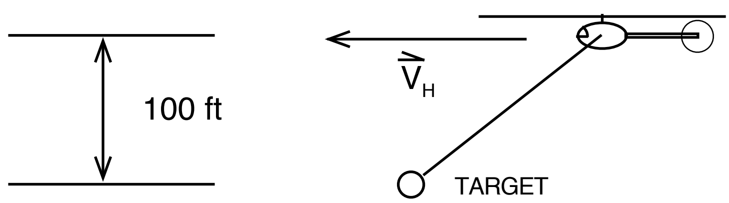

Figure 2: Vertical View of Helicopter Pass and Shot

In Figure 2, note that the shot is always taken either level or "downhill" since the helicopter pass will generally be about 100 feet above the target. Uphill shots are precluded because of the possibility of shooting the rotor blades. While the downhill angle will change the vertical "come-up" of the shot, this is no different than any other "downhill" shot. While a 100 foot altitude for a 100 yard shot results in an 18° down angle, the range is so short that this can be effectively ignored. Similarly, the "come-up" to account for the range to the target is no different than what would normally be used for the specific round and muzzle velocity were the shot taken from a stationary platform. Since most snipers zero their weapons at 100 yards, the correction for this distance is zero. The bullet is only exposed to the downward "wind" from the rotor downwash for about 20 milliseconds. In the close range cases considered here, a 35 mph downwash speed only results in about a 0.1 milli-radian required come-up, so we will ignore this small correction in the present estimation.

The goal here is the reverse lead to account for the helicopter flight motion, and we now have everything needed to complete this computation. The mean $\hat{y}$ component of the bullet velocity during the trajectory is approximately

$$\bar{V}_y = \frac{V_H \sin\theta}{2}\left(1 + e^{-\hat{C}_d X}\right) \tag{9}$$

and the reverse lead distance is given by

$$\begin{aligned} D &= \bar{V}_y\, t \\ &= \frac{V_H \sin(\theta)}{2V_B(X)\hat{C}_d}\left(e^{\hat{C}_d \frac{X_0}{\sin\theta}} - e^{-\hat{C}_d \frac{X_0}{\sin\theta}}\right) \\ &= \frac{V_H\ \sin\theta}{2\hat{C}_d(V_M + V_H\cos\theta)}\left(e^{2\hat{C}_d \frac{X_0}{\sin\theta}} - 1\right) \end{aligned} \tag{10}$$

The last form is the one to use in computation.

The reverse lead angle is

$$\psi = \arctan D/X \tag{11}$$

Typical values of the reverse lead angle ψ are 60 to 120 arc minutes, certainly larger than the field of view of most magnified scopes and probably greater than what can be dialed into the horizontal drum of a magnified scope. The use of Aim Point reflex and similar devices is recommended.

Discussion

The result

$$\boxed{D = \frac{V_H\ \sin\theta}{2\hat{C}_d(V_M + V_H\cos\theta)}\left(e^{2C_d \frac{X_0}{\sin\theta}} - 1\right)} \tag{12}$$

has some interesting properties. It is helpful to Taylor series expand the exponential term and look at the first three terms.

$$e^{2\hat{C}_d \frac{X_0}{\sin\theta}} - 1 = 2\hat{C}_d X_0 \left(\frac{1}{\sin\theta} + \frac{\hat{C}_d X_0}{\sin^2\theta} + \frac{2}{3}\frac{\left(\hat{C}_d X_0\right)^2}{\sin^3\theta} + \ldots\right) \tag{13}$$

Substitution of the Taylor series for the exponential in the reverse lead result gives

$$D = \frac{V_H}{V_M + V_H\ \cos\theta} X_0 \left(1 + \frac{\hat{C}_d}{\sin\theta} X_0 + \frac{2}{3}\frac{\hat{C}_d^2}{\sin^2\theta} X_0^2 + \ldots\right) \tag{14}$$

Two important observations may be made by inspection of this expansion. First, the drag coefficient does not appear in the first term at all. Second, note that for a common rifle round muzzle velocity of 2700 fps and a common (for this application) helicopter speed of 50 kt (about 84 fps), V_M is about 30 times larger than V_H and $\cos\theta$ is less than one. Thus, the dominant term in the reverse lead distance equation is essentially independent of both the firing angle and the ballistic coefficient. *The simple interpretation of the dominant term is that reverse lead (lag) is the ratio of the vehicle speed to the bullet muzzle speed times the range.* Also, the drag coefficient for typical rifle bullets is on the order of 10^{-4} and the minimum ranges are less than 10^3 feet, so the series converges somewhat more than one order of magnitude per term and most of the firing angle dependence comes from the second term in the expansion. Finally, note that, at small angles $\sin^n\theta$ will rapidly approach zero and the $1/\sin^n\theta$ terms will begin to dominate, making the reverse lead extremely large. This fact strongly indicates that the shooter should not attempt to shoot too early or too late in the pass. Firing prior to attaining an angle of at least 30° at close range targets or as much as 70° at longer range targets would be highly unlikely to achieve desired results.

Using this Taylor series expansion for the reverse lead and recalling $X = \frac{X_0}{\sin\theta}$, we get a similar expansion for the tangent of the reverse lead angle.

$$\psi \approx \tan\psi = D/X = \frac{V_H \sin\theta}{V_M + V_H \cos\theta}\left(1 + \frac{\hat{C}_d}{\sin\theta}X_0 + \frac{2}{3}\frac{\hat{C}_d^2}{\sin^2\theta}X_0^2 + \dots\right) \tag{15}$$

This result is also interesting because it shows that the dominant first term of the tangent of the lag angle is independent of the range as well as the drag coefficient (i.e., the Ballistic Coefficient). Again, given that the ranges are constrained to 300 yards or less, each of the succeeding terms provides a further refinement of about 10% of the preceding term. The resulting angles are small (generally less than 2 degrees), so we can use this result as a good approximation of the angle itself (in radians).

These observations become even more obvious when $\theta = 90°$ (i.e., shooting perpendicular to the helicopter's direction of motion). The reverse lead equation collapses to

$$\begin{aligned} D &= \frac{V_H}{2\hat{C}_d V_M}\left(e^{\,2\hat{C}_d X} - 1\right) \\ &= \frac{V_H}{V_M}X_0\left(1 + \hat{C}_d X_0 + \frac{2}{3}\left(\hat{C}_d X_0\right)^2 + \dots\right) \end{aligned} \tag{16}$$

and the equation for the lag angle becomes

$$\psi = \arctan\left(\frac{V_H}{V_M}\left(1 + \hat{C}_d X_0 + \frac{2}{3}\left(\hat{C}_d X_0\right)^2 + \dots\right)\right) \tag{17}$$

Because the fraction V_H/V_M is small (typically around 1/30) and, for small argument, $\arctan(x) \approx x$, we can comfortably approximate the lag angle for this case as

$$\psi \approx \frac{V_H}{V_M}\left(1 + \hat{C}_d X_0 + \frac{2}{3}\left(\hat{C}_d X_0\right)^2 + \dots\right) \tag{18}$$

The rest verifies intuition: double the vehicle speed, double the lead; half the vehicle speed, half the lead; increase the muzzle velocity, decrease the lead.

A Good Operational Technique Based on Solid Mathematical Analysis

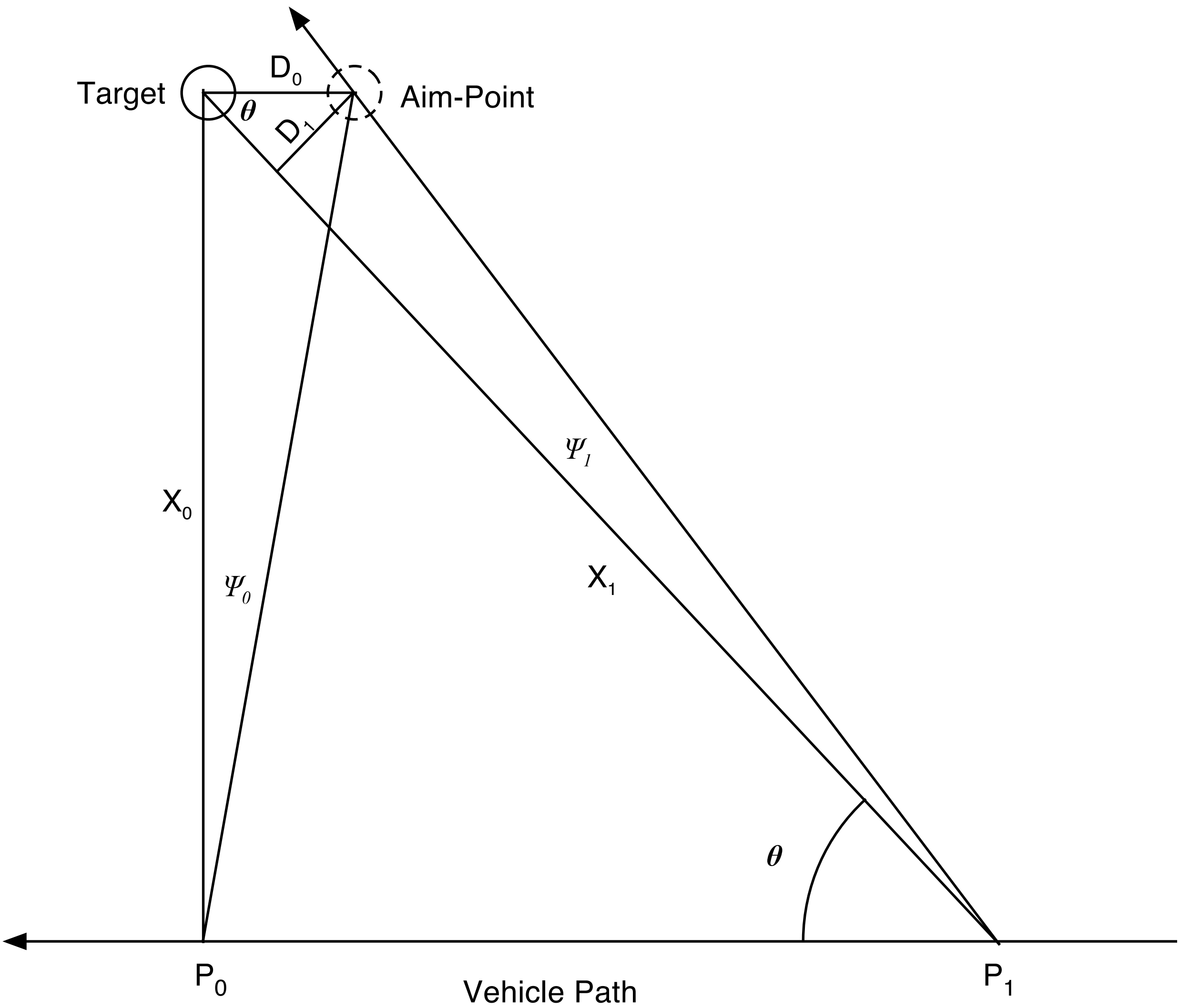

Figure 3: Geometry of Vehicle Pass and Shot

In Figure 3, the vehicle is passing horizontally from right to left along the line at the bottom of the map. The target is located at the solid circle at the top of the figure. D_0 is the reverse lead (lag) for the shot when the vehicle is at point P_0 and θ is 90°. We place a secondary object at the distance D_0 parallel to but in the opposite direction of the shooting platform vehicle path. This represents the "Aim-Point" or "virtual target" where the shooter should aim in order to hit the target. The Aim-Point is represented by the dashed circle on Figure 3. The reverse lead angle required for this shot is the angle ψ_0 in the figure.

Now, suppose the vehicle is located at some other location, P_1 along the path, such that the angle θ from the shooter to the target is less than 90°. The range from the shooter to the target is then X_1. This shot requires a different reverse lead distance, D_1, and a different reverse lead angle, ψ_1. Notice that the angle between X_1 and D_0 is also θ because the D_0 line is parallel to the vehicle path. This suggests that D_1 might be approximated by $D_0 \sin\theta$, and further suggests that, at least for some range of θ, the shooter can *always* fire at the Aim-Point (virtual target) and hit the intended target. Figure 4 illustrates the close fit of ψ by $tan^{-1}(D_0 \sin\theta/X_0)$.

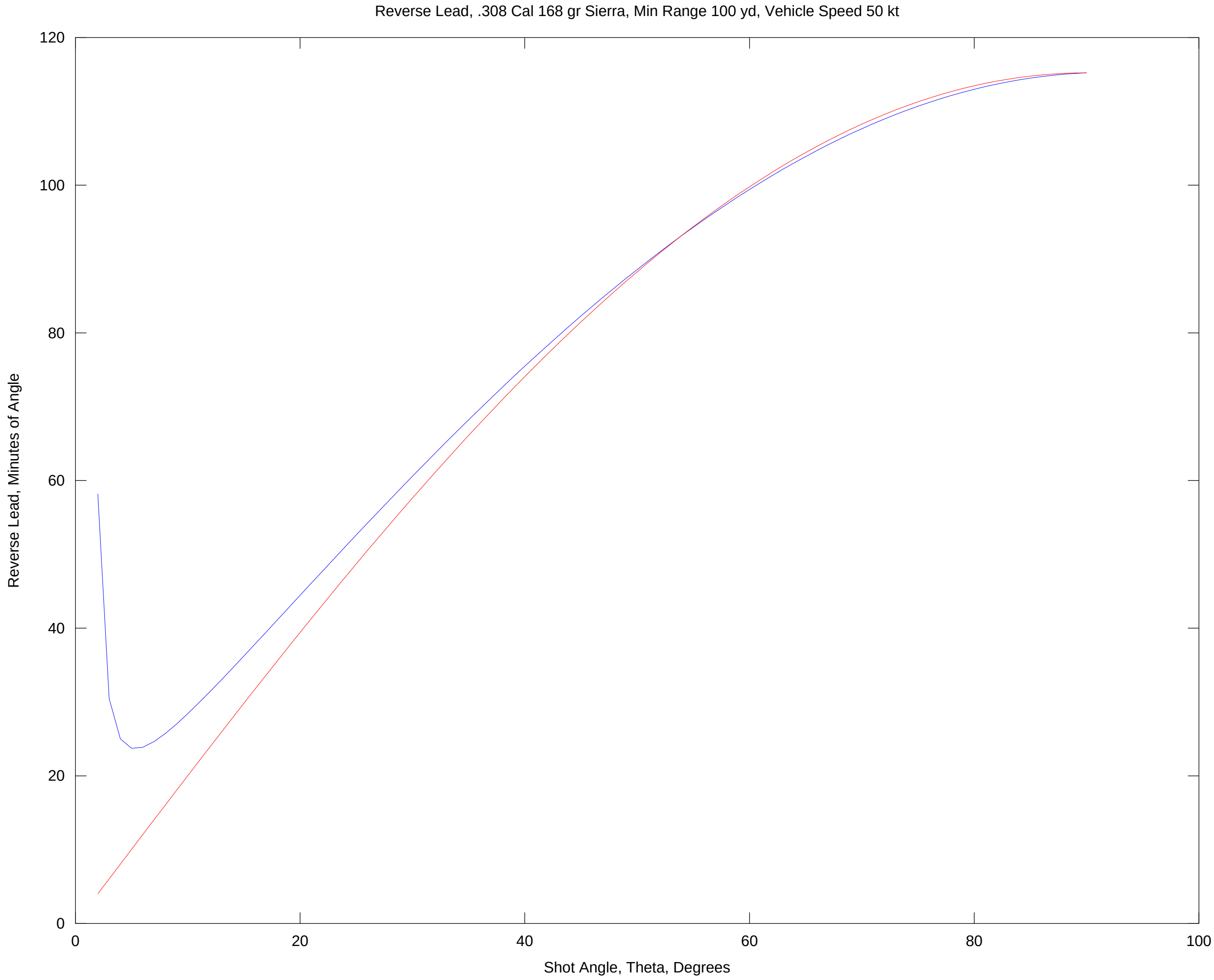

Figure 4: Comparison of actual ψ (Blue) and $tan^{-1}(D_0 \sin\theta/X_0)$ (Red)

For the case generating the ψ curve (Blue) in Figure 4, the minimum range for the pass is 100 yards. The error between the actual required reverse lead ψ and the values generated by $tan^{-1}(D_0 \sin\theta/X_0)$ at the point where $\theta = 30°$ is about 2 MOA. This is probably acceptable for this range. From $\theta = 45°$ on up, the error is essentially nil. Note that the geometric relation holds for angles beyond $\theta = 90°$ when the shots are taken as the vehicle is going away from the target.

As the range increases, the error between $tan^{-1}(D_0 \sin\theta/X_0)$ and the actual required ψ becomes unacceptable at lower θ angles. At a range of 300 yards, the error is unacceptable for angles less than about $\theta = 50°$. Still, this allows a shooter plenty of time to shoot multiple shots during the pass, particularly as shots may still be accurately taken until the angle limit is reached on the "other side" of the target (i.e., when the vehicle has passed the target).

So the operational method is: For a given ballistic coefficient, muzzle velocity, minimum pass range and vehicle speed, a reverse lead D_0 can be computed for the $\theta = 90°$, closest point on the proposed pass using either form of equation 16. In a pinch, D_0 for most rifle rounds can be estimated by just $1.11(V_H/V_M)X_0$ and the shots taken when the angle to the target is relatively close (say, within 30° either side) to perpendicular. The shooter should imagine an Aim-Point (a virtual target) at a distance D_0 to the left or right of the intended target, as appropriate. The shooter aims at that Aim-Point throughout the run and can begin firing for accuracy when the angle to the target becomes greater than 30° for closer targets or greater than 50° for farther targets. Pilots in communication with the shooter can be of great value in calling off estimated angles to the target. Note, again, that the solutions derived here work just as well for the portion of the pass *after* closest approach when the vehicle is going away from the target. That is, the shooter can still shoot at the Aim-Point after closest approach and expect the same level of effect until the angle to the target becomes too small for the method to work (30° for closer targets).

We constructed a computer program to compute the reverse leads for various angles and ranges for any weapon given the vehicle speed, the bullet muzzle velocity and associated ballistic coefficient. Results from these computations indicate that, for closest pass ranges less than 300 yards, the 77 grain .223 caliber round is probably the optimal choice because it is lighter in weight and the reverse lead results for it are almost identical to the results for the 168 grain .308. The decreased recoil of the lighter round allows the shooter shorter recoil recovery time for multiple shots, should they be necessary. This, of course, assumes that the goal is to stop a human adversary. All of the above applies just as well to an anti-material weapon such as a .50 caliber rifle.

Experimental Verification

We successfully tested the theoretical results presented here using several .223 rifles with the 55 grain Sierra hollow point round and a 0.308 rifle using the Hornady 168 grain TAP round (Sierra 168 grain BTHP Matchking bullet). Several shooters on multiple flights flew in a helicopter past an 18 inch wide steel target 100 yards from the line of flight at approximately 50 knots. To facilitate training and practice, a 24 inch diameter blue plastic drum was placed at the computed Aim-Point for the range, ballistic coefficients, approximate muzzle velocities and the helicopter speed. Shooters were instructed to shoot at the blue drum. All the shooters succeeded in hitting the steel target at least once (out of three to six separate shots taken) on their *first* run. Misses were never more than six to ten inches to either side of the target. The blue drum was *never* hit. After a few practice passes, some shooters were able to hit the target with as many as six separate shots on a single pass without even one miss.

We note again that the estimators for lead (equations 12 or 16) presented in this treatise will work just as well when shooting at a stationary target from a moving ground vehicle or boat as it does for the helicopter case we have tested. Extending the solution presented here to longer ranges requires the use of more of the (highly non-linear) drag function for the bullet and numerical integration techniques. This has been accomplished within the Empyreal Sciences, LLC "AIM-E" Sniper Assistant System.

Error Analysis

Errors in estimation of the minimum pass range, X_0, the helicopter speed, V_H, and the muzzle velocity of the bullet, V_M, are all to be expected. How much will errors in each of these estimated values effect the accuracy of shot placement?

- X_0 Using a helicopter pass speed of 50 knots, errors in the minimum range to the target will result in approximately 3% lateral errors at the target. That is, for each yard closer or farther away the minimum distance from the target to the line of the helicopter pass is, there will be a 1 inch "windage" error at the target. Errors closer to the target than intended will result in errors in the direction of motion of the helicopter with respect to the target.

V_H Assuming a 100 yard minimum range and a 2700 fps muzzle velocity, each one (1) knot of error in helicopter *ground* speed will result in 6.75 inches of "windage" error at the target. Higher than planned helicopter speeds will result in errors in the direction of helicopter motion.

V_M Assuming the helicopter speed is 50 knots and the minimum pass range is 100 yards, an error of 25 fps in the muzzle velocity will result in a "windage" error of 1 inch at the target. Muzzle velocities higher than estimated will result in errors in the opposite direction from that of the helicopter motion with respect to the target.

References

[McCoy99] McCoy, R. L.,*Modern Exterior Ballistics*, Schiffer Military History,Atglen, PA, USA, 1999